

Frobenius circulant graphs of valency six, Eisenstein-Jacobi networks, and hexagonal meshes

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Abstract

A finite Frobenius group is a permutation group which is transitive but not regular such that only the identity element can fix two points. Such a group can be expressed as a semidirect product $G = K \rtimes H$, where K is a nilpotent normal subgroup. A first-kind G -Frobenius graph is a Cayley graph on K whose connection set is an H -orbit S on K that generates K , where H is of even order or S consists of involutions. In this paper we classify all 6-valent first-kind Frobenius circulant graphs such that the underlying kernel K is cyclic. We give optimal gossiping, routing and broadcasting algorithms for such circulants and compute their forwarding indices, Wiener indices and minimum gossip time. We also prove that the broadcasting time of such a circulant is equal to its diameter plus two or three, indicating that it is efficient for broadcasting. We prove that all 6-valent first-kind Frobenius circulants with cyclic kernels are Eisenstein-Jacobi graphs, the latter being Cayley graphs on quotient rings of the ring of Eisenstein-Jacobi integers. We also prove that larger Eisenstein-Jacobi graphs can be constructed from smaller ones as topological covers, and a similar result holds for the family of 6-valent first-kind Frobenius circulants. As a corollary we show that any Eisenstein-Jacobi graph with order congruent to 1 modulo 6 and underlying Eisenstein-Jacobi integer not an associate of a real integer, is a cover of a 6-valent first-kind Frobenius circulant. We notice that a distributed real-time computing architecture known as HARTS or hexagonal mesh is a special 6-valent first-kind Frobenius circulant.

Key words: Frobenius graph; circulant graph; gossiping; routing; broadcasting; Wiener index; Eisenstein-Jacobi graph; HARTS; hexagonal mesh

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1 Introduction

A. Introduction. Searching for ‘good’ graphs as models for interconnection networks is an ongoing endeavor in theoretical computer science and network design. It is generally believed that Cayley graphs are suitable network structures due to their many attractive properties (see e.g. [1, 3, 7, 17, 23]). In fact, a number of important network topologies [17, 23] such as rings, hypercubes, cube-connected graphs, multi-loop networks, butterfly graphs, Knödel graphs [14], etc. are Cayley graphs.

Since the class of Cayley graphs is huge, one may naturally ask which Cayley graphs we should choose in order to achieve high performance. Of course, the answer to this question depends on how we measure the performance of a network, e.g. small diameter, small degree (valency), high

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connectivity, efficient data transmission, and so on. It has been proved that, as far as routing and gossiping are concerned, a large class of arc-transitive Cayley graphs, called first-kind Frobenius graphs, are ‘perfect’ in the sense that they achieve the smallest possible forwarding indices [13, 32, 37] and gossiping time [37], and possess several other attractive routing and gossiping properties [37]. (The reader is referred to [5, 16, 19, 20] and §5-6 for definitions on routing, gossiping and broadcasting.) Because of this and the importance of circulant graphs in network design [4, 21], it would be desirable [33] to classify all Frobenius circulant graphs and study their behaviours in information communication. In this paper we will classify all 6-valent first-kind Frobenius circulants such that the kernels of the underlying Frobenius groups are cyclic, and study gossiping, routing and broadcasting in such graphs. We will also study the related family of Eisenstein-Jacobi graphs [15, 26] and reveal intimate connections between such graphs and 6-valent first-kind Frobenius circulants.

The reader is referred to [33] for a classification of 4-valent first-kind Frobenius circulants and [12] for a recent study on second-kind Frobenius graphs.

B. Cayley graphs and circulants. Given a group K and an inverse-closed subset S of $K \setminus \{1\}$ (where 1 is the identity element of K), the *Cayley graph* $\text{Cay}(K, S)$ on K with respect to the *connection set* S is defined to have vertex set K such that $x, y \in K$ are adjacent if and only if $xy^{-1} \in S$. A *complete rotation* [5, 17, 18] of $\text{Cay}(K, S)$ is an automorphism of K which fixes S setwise and induces a cyclic permutation on S ; and $\text{Cay}(K, S)$ is *rotational* if it admits a complete rotation.

A Cayley graph on a cyclic group of order at least three is called a *circulant graph* or simply a *circulant*. In computer science, circulants are also called multi-loop networks [4, 21]. Let $n \geq 7$ and a, b, c be integers such that $1 \leq a, b, c \leq n - 1$ and $a, b, c, n - a, n - b, n - c$ are pairwise distinct. Then

$$TL_n(a, b, c) = \text{Cay}(\mathbb{Z}_n, S), \quad S = \{\pm[a], \pm[b], \pm[c]\}$$

is a *6-valent* circulant, that is, every vertex has degree 6. In the case when $a' + b' + c' \equiv 0 \pmod n$ for some $a' \in \{a, n - a\}, b' \in \{b, n - b\}, c' \in \{c, n - c\}$, $TL_n(a, b, c)$ is said to be *geometric*. It is so called because in this case $TL_n(a, b, c)$ can be represented [35] by a plane tessellation of hexagons.

In this paper we always assume that $TL_n(a, b, c)$ is connected, which occurs if and only if $\gcd(a, b, c, n) = 1$. In this case, $TL_n(a, b, c)$ contains a Hamilton cycle. (In fact, any Cayley graph on an Abelian group with more than two vertices is Hamiltonian; see [8].) By relabelling the vertices along a Hamilton cycle, we see that $TL_n(a, b, c)$ is isomorphic to some $TL(a', b', 1)$. Thus without loss of generality we may always assume $c = 1$ in $TL_n(a, b, c)$.

C. Transitive groups and Frobenius graphs. A group G is said to *act* on a set Ω if every $(\alpha, g) \in \Omega \times G$ corresponds to some $\alpha^g \in \Omega$ such that $\alpha^1 = \alpha$ and $(\alpha^g)^{g'} = \alpha^{gg'}$, where 1 is the identity element of G . The *stabiliser* of $\alpha \in \Omega$ in G is the subgroup $G_\alpha = \{g \in G : \alpha^g = \alpha\}$ of G , where α^g is the image of α under g . The *G -orbit* containing α is defined to be $\alpha^G = \{\alpha^g : g \in G\}$. If $G_\alpha = \{1\}$ for all $\alpha \in \Omega$, then G is called [9] *semiregular* on Ω . If $\alpha^G = \Omega$ for some (and hence all) $\alpha \in \Omega$, then G is *transitive* on Ω . If G is both transitive and semiregular on Ω , then it is said to be *regular* on Ω .

A transitive group G on Ω is called a *Frobenius group* [9] if G is not regular on Ω , and only the identity element of G can fix two points of Ω . It is well known (see e.g. [9, Section 3.4]) that a finite Frobenius group G has a nilpotent normal subgroup K , called the *Frobenius kernel* of G , which is regular on Ω . Hence $G = K \rtimes H$ (semidirect product of K by H), where H is the stabiliser of a point of Ω and is called a *Frobenius complement* of K in G . Since K is regular

on Ω , we may identify Ω with K in such a way that K acts on itself by right multiplication, and we choose H to be the stabiliser of 1 so that H acts on K by conjugation. For $x \in K$, let $x^H = \{h^{-1}xh : h \in H\}$. A *G-Frobenius graph* [13] is a Cayley graph $\Gamma = \text{Cay}(K, S)$ on K , where for some $a \in K$ satisfying $\langle a^H \rangle = K$, $S = a^H$ if $|H|$ is even or a is an involution, and $S = a^H \cup (a^{-1})^H$ otherwise. In these two cases, we call Γ a *first* or *second-kind* [37] Frobenius graph respectively. Since $\langle a^H \rangle = K$, Γ is connected in both cases.

The reader is referred to [9] and [22, 30] respectively for group- and number-theoretic terminology used in this paper.

D. Convention. It is possible for a circulant graph to be a Cayley graph on two non-isomorphic groups [29]. Thus it may happen, though not often, that for a first-kind Frobenius circulant the underlying Frobenius kernel is not isomorphic to a cyclic group. *We will only consider first-kind Frobenius circulants with cyclic underlying Frobenius kernels, but for brevity we may not mention this condition explicitly.*

E. Main results. The following is a brief summary of our main results.

- In §2 we will prove (Theorem 2) that there exists a 6-valent first-kind Frobenius circulant of order n if and only if $n \equiv 1 \pmod{6}$ and the congruence equation $x^2 - x + 1 \equiv 0 \pmod{n}$ is solvable. Moreover, under these conditions there are precisely 2^{l-1} pairwise non-isomorphic such circulants, all of which are arc-transitive and can be constructed from solutions to this congruence equation, where l is the number of distinct prime factors of n .
- In §3 we will prove (Theorem 5) that 6-valent first-kind Frobenius circulants are exactly Eisenstein-Jacobi graphs (EJ graphs for short) $EJ_{a+b\rho}$ with $\gcd(a, b) = 1$ whose order is congruent to 1 modulo 6, where $EJ_{a+b\rho}$ is defined [15, 26] (see §3) to be the Cayley graph on the additive group of $\mathbb{Z}[\rho]/(a+b\rho)$ with respect to the connection set $\{\pm 1/(a+b\rho), \pm \rho/(a+b\rho), \pm \rho^2/(a+b\rho)\}$, where $\rho = (1 + \sqrt{-3})/2$ and $\mathbb{Z}[\rho]$ is the ring of Eisenstein-Jacobi integers. We will also prove (Theorem 7) that all EJ graphs are arc-transitive.
- We will further prove (Corollary 9) in §4 that any EJ graph $EJ_{a+b\rho}$ with order congruent to 1 modulo 6 and $a+b\rho$ not an associate of any real integer, is a topological cover of a 6-valent first-kind Frobenius circulant. In fact, we will prove a more general result (Theorem 8) which asserts that larger EJ graphs can be constructed from and are covers of smaller EJ graphs. We will also prove a similar result (Theorem 10) for the family of 6-valent first-kind Frobenius circulants.
- The importance of the subfamily of 6-valent first-kind Frobenius circulants lies in that they possess very attractive routing, gossiping and broadcasting properties which are not known to hold for other EJ graphs. These will be discussed in §5 and §6. In §5 we will give optimal gossiping and routing algorithms and compute the forwarding indices and minimum gossip time for any 6-valent first-kind Frobenius circulant (Corollary 12). A by-product is a formula for the Wiener index of any 6-valent first-kind Frobenius circulant.
- In §6 we will prove that the broadcasting time of any 6-valent first-kind Frobenius circulant is equal to its diameter plus 2 or 3 (Theorem 14), indicating that it is efficient for broadcasting as well.

F. Remarks. It is interesting to notice that three groups of researchers in three different research areas came up with the same or related families of graphs independently. The motivation of the present paper, as well as that of [37] and [33], is to construct Cayley graphs that enable

very efficient information transmission. Motivated by construction of perfect codes, in [26] Martinez, Beivide and Gabidulin introduced EJ networks; and in [15] Flahive and Bose further studied EJ networks and related Gaussian networks [27]. As mentioned above, on the one hand, 6-valent first-kind Frobenius circulants are precisely EJ graphs $EJ_{a+b\rho}$ with $\gcd(a, b) = 1$ and order congruent to 1 modulo 6. This connection enables us to compute distance distributions of 6-valent first-kind Frobenius circulants by using the corresponding results [15] for EJ graphs. On the other hand, any $EJ_{a+b\rho}$ with order congruent to 1 modulo 6 and $a + b\rho$ not an associate of any real integer can be constructed from a 6-valent first-kind Frobenius circulant as a topological cover. This allows us to compute the forwarding indices and the minimum gossip times of such EJ graphs by using the general theory developed in [37].

The first version of this paper was made public in May 2012 (see <http://arxiv.org/pdf/1205.5877v1.pdf>). It was only recently that we found that a very special subfamily of 6-valent first-kind Frobenius circulants had physically been used [6, 11, 31] as multiprocessor interconnection networks at the Real-Time Computing Laboratory, The University of Michigan. They are called HARTS (*Hexagonal Architecture for Real-Time Systems*) [11, 31], *C-wrapped hexagonal meshes* [11], or *hexagonal mesh interconnection networks* [2]. As we will see in Example 1, they are indeed 6-valent first-kind Frobenius circulants. In the combinatorial community these circulants were first studied in [35], and their optimal routing and gossiping algorithms were given in [34].

We remark that the routing problem considered in this paper is different from that studied in [2, 6, 15, 27], where routing is mainly about computing shortest paths and distance between two vertices. An optimal one-to-all communication (broadcasting) algorithm for HARTS H_k of diameter $k - 1$ was given in [6, Algorithm A2], using $k + 2$ steps when $k \geq 3$. Since H_k is a special 6-valent first-kind Frobenius circulant, Theorem 14 in the present paper can be viewed as a generalization of this result to a much larger family of graphs. In [2] an algorithm for all-to-all communication (that is, gossiping in the present paper and [34]) for H_k was given which requires $3k(k - 1)/2$ time steps. However, by [34, Theorem 4], $k(k - 1)/2$ time steps are sufficient and necessary for H_k , and an algorithm using $k(k - 1)/2$ steps was given in [34, Algorithm 2]. Yet this is a special case of a more general result: In Algorithm 2 we will give an optimal all-to-all communication algorithm for any 6-valent first-kind Frobenius circulant of order n using $(n - 1)/6$ time steps.

2 Classification of 6-valent first-kind Frobenius circulants

A. Preparations. We always use $[m]$ to denote the residue class modulo n , where n is a positive integer. Let $\mathbb{Z}_n^* = \{[m] : 1 \leq m \leq n - 1, \gcd(m, n) = 1\}$ be the multiplicative group of units of the ring \mathbb{Z}_n . We use $[m]^{-1}$ to denote the inverse element of $[m]$ in \mathbb{Z}_n^* . It is well known that $\text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. As the automorphism group of \mathbb{Z}_n , \mathbb{Z}_n^* acts on \mathbb{Z}_n by usual multiplication: $[x][m] = [xm]$, $[m] \in \mathbb{Z}_n^*$, $[x] \in \mathbb{Z}_n$. $\mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ acts on \mathbb{Z}_n such that $[x]^{([y], [m])} = [(x + y)m]$ for $[x], [y] \in \mathbb{Z}_n$ and $[m] \in \mathbb{Z}_n^*$.

Lemma 1. ([33, Lemma 4]) *A subgroup H of \mathbb{Z}_n^* is semiregular on $\mathbb{Z}_n \setminus \{[0]\}$ if and only if $[h - 1] \in \mathbb{Z}_n^*$ for all $[h] \in H \setminus \{[1]\}$.*

For an odd prime p and an integer r , the Legendre symbol $\left(\frac{r}{p}\right)$ is defined [22, 30] to be 1 if r is a quadratic residue modulo p , -1 if r is a quadratic nonresidue modulo p , and 0 if p divides r .

A graph Γ is called *G-arc-transitive* if $G \leq \text{Aut}(\Gamma)$ is transitive on the set of arcs of Γ , where an *arc* is an ordered pair of adjacent vertices.

A group K is a *connected m-CI-group* [24] if for any connected Cayley graphs $\text{Cay}(K, S)$ and $\text{Cay}(K, T)$ such that $|S| = |T| \leq m$ and $\text{Cay}(K, S) \cong \text{Cay}(K, T)$, there exists $\sigma \in \text{Aut}(K)$ such that $T = S^\sigma$.

B. Classification. The following is the main result in this section.

Theorem 2. *Let $n \geq 7$ be an integer. The following statements are equivalent:*

- (a) *there exists a 6-valent first-kind Frobenius circulant $TL_n(a, b, 1)$ of order n such that the kernel of the underlying Frobenius group is cyclic;*
- (b) *$n \equiv 1 \pmod{6}$ and the following congruence equation has a solution:*

$$x^2 - x + 1 \equiv 0 \pmod{n}. \quad (1)$$

Moreover, if one of these conditions is satisfied, then

- (c) *each prime factor of n is congruent to 1 modulo 6;*
- (d) *each solution a to (1) gives rise to a 6-valent first-kind Frobenius circulant $TL_n(a, b, 1)$, and vice versa; in this case we have $b \equiv a - 1 \pmod{n}$ and $TL_n(a, a - 1, 1)$ is a rotational, geometric, $\mathbb{Z}_n \rtimes H$ -arc-transitive and first-kind $\mathbb{Z}_n \rtimes H$ -Frobenius graph admitting $[a]$ and $-[a^2]$ as complete rotations, where*

$$H = \langle [a] \rangle = \{\pm[1], \pm[a], \pm[a^2]\} = \{\pm[1], \pm[a], \pm[a - 1]\} \leq \mathbb{Z}_n^*; \quad (2)$$

- (e) *there are exactly 2^{l-1} pairwise non-isomorphic 6-valent first-kind Frobenius circulants of order n , where l is the number of distinct prime factors of n , and each of them is isomorphic to $TL_n(a, a - 1, 1)$ for some a as above.*

Proof (i) Suppose first that there exists a first-kind Frobenius circulant $TL_n(a, b, 1)$ of order n such that the kernel of the underlying Frobenius group is cyclic. Then there exists a subgroup H of \mathbb{Z}_n^* such that $|H| = 6$, $\mathbb{Z}_n \rtimes H$ is a Frobenius group and $TL_n(a, b, 1)$ is a first-kind $\mathbb{Z}_n \rtimes H$ -Frobenius circulant. Thus H is semiregular on $\mathbb{Z}_n \setminus \{[0]\}$ and so $n \equiv 1 \pmod{6}$. Moreover, $S = \{\pm[1], \pm[a], \pm[b]\}$ is an H -orbit on \mathbb{Z}_n and hence H is regular on S . Since $[1] \in S$, it follows that $S = H$. Since H is Abelian with $|H| = 6$, it must be a cyclic group of order 6. So we may assume $H = \langle [h] \rangle = \{[1], [h], [h^2], [h^3], [h^4], [h^5]\}$ for an element $[h]$ of \mathbb{Z}_n^* with order 6. Since $S = H$, there exists $1 \leq i \leq 5$ such that $[h^i] = -[1]$. Hence $[h^{2i}] = [1]$ and so 6 divides $2i$. Therefore, $i = 3$, $[h^3] = -[1]$ and $H = \{\pm[1], \pm[h], \pm[h^2]\}$. Since $S = H$, without loss of generality we may assume $[a] = [h]$, so that $a^3 + 1 \equiv 0 \pmod{n}$, $b \equiv a^2 \pmod{n}$ and $H = \{\pm[1], \pm[a], \pm[a^2]\}$. Since H is semiregular on $\mathbb{Z}_n \setminus \{[0]\}$, by Lemma 1 we have $[-a - 1] \in \mathbb{Z}_n^*$ and hence $\gcd(a + 1, n) = 1$. Since $a^3 + 1 = (a + 1)(a^2 - a + 1) \equiv 0 \pmod{n}$, a is a solution to (1). Hence $b \equiv a^2 \equiv a - 1 \pmod{n}$ and $TL_n(a, b, 1) = TL_n(a, a - 1, 1) = TL_n(n - a, a - 1, 1)$ is geometric. It is readily seen that $[a]$ and $-[a^2]$ are complete rotations of $TL_n(a, a - 1, 1)$. Hence $TL_n(a, a - 1, 1)$ is rotational as well. Moreover, $TL_n(a, a - 1, 1)$ is $\mathbb{Z}_n \rtimes H$ -arc-transitive by [37, Lemma 2.1].

(ii) Now suppose $n \equiv 1 \pmod{6}$ and (1) is solvable. Write $n = p_1^{e_1} p_2^{e_2} \dots p_l^{e_l}$, where $p_1, p_2, \dots, p_l \geq 5$ are distinct primes and $e_1, e_2, \dots, e_l \geq 1$ are integers. Let a be a solution to (1). We first prove $\gcd(a + 1, n) = 1$. Suppose otherwise. Then $\gcd(2a - 1, n) > 1$

as $a(a+1) \equiv 2a-1 \pmod n$ by (1). Without loss of generality we may assume that p_1 divides $\gcd(2a-1, n)$. Since a is a solution to (1), by [30, Section 2.5] it is also a solution to $x^2 - x + 1 \equiv 0 \pmod{p_1^{e_1}}$. Since n is odd, it follows that a is a solution to $4x^2 - 4x + 4 \equiv 0 \pmod{p_1^{e_1}}$, that is, $(2x-1)^2 \equiv -3 \pmod{p_1^{e_1}}$. Moreover, $a = (p_1^{e_1} + 1)(v+1)/2 \pmod{p_1^{e_1}}$ for some integer v satisfying $v^2 \equiv -3 \pmod{p_1^{e_1}}$. Thus $2a-1 \equiv v \pmod{p_1^{e_1}}$ and so p_1 divides v . Since p_1 divides $v^2 + 3$, it follows that p_1 divides 3, which contradicts the fact $p_1 \geq 5$. Therefore, any solution a to (1) satisfies $\gcd(a+1, n) = 1$.

Since a satisfies (1), it also satisfies $a^3 + 1 \equiv 0 \pmod n$. Hence $\gcd(a, n) = 1$ and $H := \langle [a] \rangle = \{\pm[1], \pm[a], \pm[a^2]\} \leq \mathbb{Z}_n^*$. Since $a^3 \equiv -1 \pmod n$, $a^6 \equiv 1 \pmod n$ and so the order of $[a]$ in \mathbb{Z}_n^* is a divisor of 6. Obviously, $a^3 \not\equiv 1 \pmod n$. If $a^2 \equiv 1 \pmod n$, then since $\gcd(a+1, n) = 1$ we have $a \equiv 1 \pmod n$, which is a contradiction. Thus $[a]$ must have order 6 in \mathbb{Z}_n^* and hence $|H| = 6$. Since a satisfies (1), it follows that $\gcd(a^2 + 1, n) = \gcd(a, n) = 1$ and $\gcd(a-1, n) = \gcd(a^2, n) = 1$. Since $\gcd(a+1, n) = 1$, we have $\gcd(a^2 - 1, n) = 1$. Therefore, H is semiregular on $\mathbb{Z}_n \setminus \{[0]\}$ by Lemma 1. Hence $\mathbb{Z}_n \rtimes H$ is a Frobenius group. Set $S := H$. Then S is an H -orbit on $\mathbb{Z}_n \setminus \{[0]\}$ and $\langle S \rangle = \mathbb{Z}_n$ since $[1] \in S$. Hence $\text{Cay}(\mathbb{Z}_n, S) = \text{TL}_n(a, b, 1)$ is a 6-valent first-kind $\mathbb{Z}_n \rtimes H$ -Frobenius graph, where $b \equiv a^2 \equiv a-1 \pmod n$ and obviously the kernel of $\mathbb{Z}_n \rtimes H$ is cyclic.

(iii) Suppose $n \equiv 1 \pmod 6$ and (1) is solvable. Then all statements in (d) are true by the proof above. Moreover, by [30, Section 2.5], a is a solution to (1) if and only if it is a solution to the set of congruence equations:

$$x^2 - x + 1 \equiv 0 \pmod{p_i^{e_i}}, \quad i = 1, 2, \dots, l. \quad (3)$$

Since 4 is co-prime to p_i , we have: a is a solution to (1) $\Leftrightarrow a$ is a solution to $4x^2 - 4x + 4 \equiv 0 \pmod{p_i^{e_i}}$, i.e. $(2x-1)^2 \equiv -3 \pmod{p_i^{e_i}}$ for $i = 1, 2, \dots, l \Leftrightarrow a$ is a solution to $2x-1 \equiv v \pmod{p_i^{e_i}}$, for $i = 1, 2, \dots, l$, where v is a solution to $x^2 \equiv -3 \pmod{p_i^{e_i}}$ for each i . Since p_i is odd, for any integer v , $2x-1 \equiv v \pmod{p_i^{e_i}}$ has a unique solution, namely $(p_i^{e_i} + 1)(v+1)/2 \pmod{p_i^{e_i}}$. Since $\gcd((p_i^{e_i} + 1)/2, p_i^{e_i}) = \gcd(2, p_i^{e_i}) = 1$, the solutions of $x^2 - x + 1 \equiv 0 \pmod{p_i^{e_i}}$ and that of $x^2 \equiv -3 \pmod{p_i^{e_i}}$ are in one-to-one correspondence.

Since (1) is solvable by our assumption, from the argument above $x^2 \equiv -3 \pmod{p_i^{e_i}}$ is solvable for each i . By [22, Proposition 4.2.3] and the fact $p_i \geq 5$, this implies that $x^2 \equiv -3 \pmod{p_i}$ is solvable, that is, $\left(\frac{-3}{p_i}\right) = 1$ for each i . Since $\left(\frac{-3}{p_i}\right) = (-1)^{\frac{p_i-1}{2}} \left(\frac{3}{p_i}\right)$, it follows that either $p_i \equiv 1 \pmod 4$ and $\left(\frac{3}{p_i}\right) = 1$ or $p_i \equiv -1 \pmod 4$ and $\left(\frac{3}{p_i}\right) = -1$. By [22, Theorem 2, Chapter 5], $\left(\frac{3}{p_i}\right) = 1$ if and only if $p_i \equiv \pm d^2 \pmod{12}$, where d is an odd integer co-prime to 3. Thus, $\left(\frac{3}{p_i}\right) = 1$ if and only if $p_i \equiv \pm 1 \pmod{12}$, and $\left(\frac{3}{p_i}\right) = -1$ if and only if $p_i \equiv \pm 5 \pmod{12}$. Therefore, $p_i \equiv 1 \pmod 6$ for $i = 1, 2, \dots, l$ and (c) holds.

Let $N(n)$ denote the number of solutions to (1) and $\hat{N}(n)$ the number of solutions to $x^2 \equiv -3 \pmod n$. Then $N(n) = \prod_{i=1}^l N(p_i^{e_i})$ and $\hat{N}(n) = \prod_{i=1}^l \hat{N}(p_i^{e_i})$ by [30, Theorem 2.18]. Note that $N(p_i^{e_i}) = \hat{N}(p_i^{e_i})$ by our discussion above. Hence $N(n) = \hat{N}(n)$. Since $p_i \geq 5$ is not a divisor of 2 or -3 , by [22, Proposition 4.2.3] we have $\hat{N}(p_i^{e_i}) = N(p_i)$ for each i . Since $p_i \equiv 1 \pmod 6$ as proved above, we have $\hat{N}(p_i) \geq 1$ as shown earlier. Thus, by [30, Corollary 2.28], $\hat{N}(p_i) = 2$ and hence $N(n) = \hat{N}(n) = \prod_{i=1}^l \hat{N}(p_i^{e_i}) = 2^l$. For each solution a to (1), we have $(-a)^2 - (-a) + 1 \equiv 2a \not\equiv 0 \pmod n$, $(a^2)^2 - a^2 + 1 \equiv -a^2 - a + 1 \equiv -2(a-1) \not\equiv 0 \pmod n$ and $(-a^2)^2 - (-a^2) + 1 \equiv a^2 - a + 1 \equiv 0 \pmod n$. Hence $-a^2$ is also a solution to (1) and moreover no residue class in $S = \{\pm[1], \pm[a], \pm[a^2]\}$ other than $[a]$ and $-[a^2]$ is a solution to (1). It is proved in [24, Theorem 4.2] that any Abelian group G is a connected p -CI group,

where p is the least prime factor of $|G|$. Applying this to \mathbb{Z}_n and noting that all $p_i \geq 7$, it follows that \mathbb{Z}_n is a connected 7-CI group. Therefore, if $TL_n(a_1, b_1, 1) \cong TL_n(a_2, b_2, 1)$ for two solutions a_1, a_2 to (1) (where $b_1 \equiv a_1^2, b_2 \equiv a_2^2 \pmod{n}$), then there exists $[m] \in \mathbb{Z}_n^*$ such that $S_1[m] = S_2$, where $S_1 = \{\pm[1], \pm[a_1], \pm[a_1^2]\}$ and $S_2 = \{\pm[1], \pm[a_2], \pm[a_2^2]\}$. Since $[1] \in S_2$, there exists $[x] \in S_1$ such that $[xm] = [1]$. Since S_1 is a subgroup of \mathbb{Z}_n^* , this implies $[m] \in S_1$ and consequently $S_2 = S_1[m] = S_1$. Note that the solutions a and $-a^2$ to (1) give rise to the same graph. Therefore, there are exactly 2^{l-1} pairwise non-isomorphic 6-valent first-kind Frobenius circulants of order n . \square

C. Convention and remarks. Whenever we mention a 6-valent first-kind Frobenius circulant $TL_n(a, a-1, 1)$ we assume without mentioning that it is as in Theorem 2 so that the kernel of the underlying Frobenius group is isomorphic to \mathbb{Z}_n .

Remark 1. (a) Note that the connection set of $TL_n(a, a-1, 1)$ in Theorem 2 is the same as the complement H of the underlying Frobenius group.

(b) Since $\gcd(a, n) = \gcd(a-1, n) = \gcd(1, n) = 1$, $TL_n(a, a-1, 1)$ can be decomposed into three edge-disjoint Hamilton cycles. Note that we also have $\gcd(a+1, n) = 1$ and so $\gcd(a-2, n) = 1$ as $a-2 \equiv (a-1)(a+1) \pmod{n}$. One can verify that $\gcd(2a-1, n) = 1$. These observations will be used in §4.

(c) By Theorem 2, a necessary condition for the existence of a 6-valent first-kind Frobenius circulant of order n is $\phi(n) \equiv 0 \pmod{6}$, where $\phi(n)$ is Euler's totient function.

(d) Given solutions to $x^2 \equiv -3 \pmod{p}$ as input, there exist efficient algorithms to compute solutions to $x^2 \equiv -3 \pmod{p^e}$ for any prime p and integer $e \geq 1$. Once we work out the solutions to $x^2 \equiv -3 \pmod{p_i^{e_i}}$ for all i , we can find all solutions v to $x^2 \equiv -3 \pmod{n}$ by using a standard procedure (see e.g. [30, Section 2.5]) based on the Chinese Remainder Theorem.

Using the argument after (3) and noting that n is odd, we see that a is a solution to (1) if and only if $a \equiv (n+1)(v+1)/2 \pmod{n}$ for a solution v to $x^2 \equiv -3 \pmod{n}$. So we obtain the following solution a to (1) arisen from v (the other solution arisen from v is $-a^2 \equiv n-a+1 \pmod{n}$):

$$a \equiv \begin{cases} \frac{v+1}{2}, & \text{if } v \text{ is odd} \\ \frac{n+v+1}{2}, & \text{if } v \text{ is even.} \end{cases} \quad (4)$$

In this way we can find all solutions a to (1) and hence all 6-valent first-kind Frobenius circulants $TL_n(a, a-1, 1)$ of order n . Example 1 below and Example 3 in the next section show that both cases in (4) can occur.

(e) A solution to $x^3 \equiv -1 \pmod{n}$ may not produce a 6-valent first-kind Frobenius circulant even when every prime factor of n is congruent to 1 modulo 6. For example, 12 is a solution to $x^3 \equiv -1 \pmod{91}$ but not a solution to $x^2 - x + 1 \equiv 0 \pmod{91}$. Note that $12^2 \equiv 53 \pmod{91}$ but $TL_n(1, 12, 53)$ is not a first-kind Frobenius graph, for otherwise $H = \langle [12] \rangle \leq \mathbb{Z}_{91}^*$ would be semiregular on $\mathbb{Z}_{91} \setminus \{[0]\}$, which is not true as $\gcd(52, 91) = 13 > 1$. It can be verified that $TL_n(1, 12, 53)$ is rotational but not geometric.

D. Prime power orders. Theorem 2 together with its proof implies the following result.

Corollary 3. *Let p be a prime such that $p \equiv 1 \pmod{6}$. Then for every integer $e \geq 1$ there is a unique 6-valent first-kind Frobenius circulant of order p^e , namely*

$$\Gamma(p^e) = TL_{p^e}(a_e, a_e - 1, 1),$$

where $a_e = (p^e + 1)(v + 1)/2 \pmod{p^e}$ with v a solution to $x^2 \equiv -3 \pmod{p^e}$.

This case is both interesting and significant because, as we will see in Theorem 10, every 6-valent first-kind Frobenius circulant with order a multiple of p^e is a topological cover of $\Gamma(p^e)$. In particular, every graph in the sequence $\Gamma(p), \Gamma(p^2), \dots, \Gamma(p^e), \dots$ is a topological cover of the graphs preceding it. Moreover, starting from $\Gamma(p)$ we can construct $\Gamma(p^e)$ recursively by constructing a solution a_{s+1} to $x^2 - x + 1 \equiv 0 \pmod{p^{s+1}}$ based on a solution a_s to $x^2 - x + 1 \equiv 0 \pmod{p^s}$, $s = 1, 2, \dots$. Using a standard procedure in number theory (see e.g. [30, Section 2.6]), we have $a_{s+1} = a_s + p^s t$, $s = 1, 2, \dots$, where t is a solution to $(2a_s - 1)t \equiv -(a_s^2 - a_s + 1)/p^s \pmod{p}$. Thus, beginning with a_1 , we can construct a_2 and hence $\Gamma(p^2)$. Based on a_2 we then construct a_3 and hence $\Gamma(p^3)$, and so on. For instance, in the case when $p = 7$, we get $a_1 = 3, a_2 = 31, a_3 = 325, \dots$, recursively.

E. HARTS, or hexagonal meshes. HARTS was proposed [6] as a distributed real-time computing system, and its properties were studied in [6, 11, 2]. We now explain that it belongs to the family of 6-valent first-kind Frobenius circulants.

Example 1. Let $k \geq 2$ be an integer and $n_k = 3k^2 + 3k + 1$. It was proved in [34, Theorem 1] that $TL_{n_k} = TL_{n_k}(3k + 2, 3k + 1, 1)$ ($= TL_{n_k}(1, 3k + 1, -(3k + 2))$) is a 6-valent first-kind Frobenius circulant. This is now an immediate consequence of Theorem 2, because $v = 6k + 3$ is a solution to $x^2 \equiv -3 \pmod{n}$ and it gives rise to the solution $a = (v + 1)/2 = 3k + 2$ to (1).

It is known [35] that TL_{n_k} has the maximum possible order among all 6-valent geometric circulants of diameter k . Optimal gossiping and routing schemes, and broadcasting and embedding properties of TL_{n_k} have been studied in [34] and [36], respectively.

The HARTS H_k of size k has diameter $k - 1$ and $n_{k-1} = 3k^2 - 3k + 1$ vertices [6], and is isomorphic [6, 2] to the circulant $\text{Cay}(\mathbb{Z}_{n_{k-1}}, S)$ with $S = \{\pm[k - 1], \pm[k], \pm[2k - 1]\}$, where the residue classes are modulo n_{k-1} . Since $[3k] \in \mathbb{Z}_{n_{k-1}}^*$, we have $\text{Cay}(\mathbb{Z}_{n_{k-1}}, S) \cong \text{Cay}(\mathbb{Z}_{n_{k-1}}, S')$ via the isomorphism $[x] \mapsto [3k][x]$, where $S' = \{\pm[3k][k - 1], \pm[3k][k], \pm[3k][2k - 1]\} = \{\pm[1], \pm[3k - 1], \pm[3k - 2]\}$. It follows that H_k is isomorphic to $TL_{n_{k-1}}$.

In [2] it was noted that H_k is isomorphic to the EJ graph $EJ_{k+(k-1)\rho}$. Thus $TL_{n_{k-1}}$ is isomorphic to $EJ_{k+(k-1)\rho}$. This is not a coincidence: we will see in Theorem 5 that any EJ graph $EJ_{a+b\rho}$ with $\gcd(a, b) = 1$ is isomorphic to a 6-valent first-kind Frobenius circulant. \square

3 Frobenius versus Eisenstein-Jacobi

The study of Eisenstein-Jacobi graphs was motivated by perfect code construction [26] and interconnection network design [15]. In this section we prove that 6-valent first-kind Frobenius graphs form a (proper) subfamily of the family of Eisenstein-Jacobi graphs. This result enables us to obtain the distance distribution of the former from that of the latter [15]. We will also show that all Eisenstein-Jacobi graphs are arc-transitive.

A. Eisenstein-Jacobi graphs. Let $\rho = (1 + \sqrt{-3})/2$ and let $\mathbb{Z}[\rho] = \{x + y\rho : x, y \in \mathbb{Z}\}$ be the ring of Eisenstein-Jacobi integers [22]. It is well-known [22] that $\mathbb{Z}[\rho]$ is a Euclidean domain with norm defined by $N(x + y\rho) = x^2 + xy + y^2$. We have $\rho^2 - \rho + 1 = 0, \rho^3 = -1$ and the set of units of $\mathbb{Z}[\rho]$ is $\{\rho^j : j \in \mathbb{Z}\} = \{\pm 1, \pm\rho, \pm\rho^2\} = \{\pm 1, \pm\rho, \pm(\rho - 1)\}$.

Let $0 \neq \alpha = c + d\rho \in \mathbb{Z}[\rho]$. Consider the quotient ring $\mathbb{Z}[\rho]/(\alpha)$ of $\mathbb{Z}[\rho]$ with respect to the principal ideal (α) . For any $\eta \in \mathbb{Z}[\rho]$, let $[\eta]_\alpha \in \mathbb{Z}[\rho]/(\alpha)$ denote the residue class containing η modulo α . If $N(\alpha) \geq 7$, the *Eisenstein-Jacobi graph* (or *EJ graph* for short) EJ_α generated by α is defined [26] as the Cayley graph on the additive group of $\mathbb{Z}[\rho]/(\alpha)$ with respect to $\{\pm[1]_\alpha, \pm[\rho]_\alpha, \pm[\rho^2]_\alpha\}$. More explicitly, EJ_α has vertex set $\mathbb{Z}[\rho]/(\alpha)$ such that $[\xi]_\alpha$ and $[\eta]_\alpha$ are

adjacent if and only if $[\xi]_\alpha - [\eta]_\alpha = [\rho^j]$ for some j . The assumption $N(\alpha) \geq 7$ ensures that $\pm[1]_\alpha, \pm[\rho]_\alpha, \pm[\rho^2]_\alpha$ are pairwise distinct and so EJ_α is a 6-valent graph with $N(\alpha)$ vertices.

Instead of $\mathbb{Z}[\rho]$, in [26] EJ graphs are defined on $\mathbb{Z}[\omega]$ with norm $N(x + y\omega) = x^2 - xy + y^2$, where $\omega = (-1 + \sqrt{-3})/2$. Although $EJ_{c+d\omega}$ defined on $\mathbb{Z}[\omega]$ in this way has $c^2 - cd + d^2$ vertices and is different from our graph $EJ_{c+d\rho}$, the family of EJ graphs is the same [15] no matter whether $\mathbb{Z}[\rho]$ or $\mathbb{Z}[\omega]$ is used, and all results for EJ graphs on $\mathbb{Z}[\omega]$ can be translated into results for EJ graphs on $\mathbb{Z}[\rho]$. Our terminology in this and the next sections agrees with that in [15].

Lemma 4. ([26, Theorem 20]) *Let $\alpha = c + d\rho \in \mathbb{Z}[\rho]$ be such that $N(\alpha) \geq 7$ and $\gcd(c, d) = 1$. Denote $n = N(\alpha)$. Then*

$$EJ_\alpha \cong TL_n(c, d, c + d).$$

In fact, since $\gcd(c, d) = 1$, any integer can be expressed as $dx - cy$ for some integers x and y . One can verify that

$$\mathbb{Z}_{N(\alpha)} \rightarrow \mathbb{Z}[\rho]/(\alpha), dx - cy \bmod n \mapsto [x + y\rho]_\alpha, x, y \in \mathbb{Z} \quad (5)$$

defines the required isomorphism from $TL_n(c, d, c + d)$ to EJ_α .

B. 6-valent first-kind Frobenius circulants are EJ graphs.

Theorem 5. (a) *Every 6-valent first-kind Frobenius circulant $TL_n(a, a - 1, 1)$ is isomorphic to some EJ_α with $\alpha = c + d\rho$ satisfying $\gcd(c, d) = 1$.*

Moreover, letting k be the integer defined by $a^2 - a + 1 = kn$, we have $\alpha = (rn + m) + (sn - ma)\rho$, $(rn + m) + (sn + m(a - 1))\rho$ or $(rn + ma) + (sn - m(a - 1))\rho$, where (m, r, s) is a solution to one of the following Diophantine equations, respectively,

$$km^2 - [(a - 2)r + (2a - 1)s]m + (r^2 + rs + s^2)n = 1 \quad (6)$$

$$km^2 + [(a + 1)r + (2a - 1)s]m + (r^2 + rs + s^2)n = 1 \quad (7)$$

$$km^2 + [(a + 1)r - (a - 2)s]m + (r^2 + rs + s^2)n = 1. \quad (8)$$

(b) *Let $\alpha = c + d\rho \in \mathbb{Z}[\rho]$ be such that $N(\alpha) \geq 7$ and $\gcd(c, d) = 1$. Then EJ_α is isomorphic to a 6-valent first-kind Frobenius circulant if and only if $N(\alpha) \equiv 1 \pmod{6}$.*

Proof (a) Let $TL_n(a, a - 1, 1)$ be a 6-valent first-kind Frobenius circulant, where $n \geq 7$, $n \equiv 1 \pmod{6}$, and a is a solution to (1). Define $f : \mathbb{Z}[\rho] \rightarrow \mathbb{Z}_n$ by $f(x + y\rho) = [x + ya]$. Since a satisfies $a^2 - a + 1 \equiv 0 \pmod{n}$, one can verify that f is a well-defined ring homomorphism. Since $\mathbb{Z}[\rho]$ is a Euclidean domain, it is a principal ideal domain. Thus the kernel of f must be a principal ideal of $\mathbb{Z}[\rho]$; that is, $\ker(f) = (\alpha)$ for some $0 \neq \alpha = c + d\rho \in \mathbb{Z}[\rho]$. Since $\mathbb{Z}[\rho]/(\alpha) \cong \mathbb{Z}_n$ and f maps $\{\pm 1, \pm\rho, \pm(\rho - 1)\}$ to $\{\pm[1], \pm[a], \pm[a - 1]\}$, we obtain $EJ_\alpha \cong TL_n(a, a - 1, 1)$.

Thus $N(\alpha) = n$, and by Lemma 4, $TL_n(a, a - 1, 1) \cong TL_n(c, d, c + d)$. Since \mathbb{Z}_n is a 7-CI-group by [24, Theorem 4.2], it follows that there exist an integer m with $\gcd(m, n) = 1$ such that $\{[ma], [m(a - 1)], [m], -[ma], -[m(a - 1)], -[m]\} = \{[c], [d], [c + d], -[c], -[d], -[c + d]\}$. Since $2, a - 1$ and a are all coprime to n , we have $\{[c], [d]\} = \{[m], -[ma]\}$, $\{-[m], [ma]\}$, $\{[m], [m(a - 1)]\}$, $\{-[m], -[m(a - 1)]\}$, $\{[ma], -[m(a - 1)]\}$ or $\{-[ma], [m(a - 1)]\}$. Since the roles of $[c]$ and $[d]$ are symmetric and $TL_n(c, d, c + d) = TL_n(-c, -d, -c - d)$, it suffices to consider three cases: $([c], [d]) = ([m], -[ma])$, $([m], [m(a - 1)])$ or $([ma], -[m(a - 1)])$.

In the case when $([c], [d]) = ([m], -[ma])$, there exist integers r and s such that $c = rn + m$, $d = sn - ma$, $\gcd(rn + m, sn - ma) = 1$ and $n = (rn + m)^2 + (rn + m)(sn - ma) + (sn - ma)^2$.

This is equivalent to saying that (m, r, s) is a solution to (6). One can verify that (m, r, s) satisfies $\gcd(m, n) = 1$ and $\gcd(c, d) = 1$. The other two cases can be treated similarly.

(b) Denote $n = N(\alpha) = c^2 + cd + d^2$. The necessity follows from Theorem 2 immediately.

To prove the sufficiency, suppose $n \equiv 1 \pmod{6}$. It is clear that any prime common divisor of d and n is also a divisor of c . This together with the assumption $\gcd(c, d) = 1$ implies $\gcd(d, n) = 1$. Hence $[d]$ is an element of \mathbb{Z}_n^* . Let $[g]$ be the inverse of $[d]$ in \mathbb{Z}_n^* and let $a \equiv -cg \pmod{n}$ be such that $0 \leq a \leq n-1$. Multiplying $c^2 + cd + d^2 = n$ by g^2 , we obtain $a^2 - a + 1 \equiv 0 \pmod{n}$. Thus, by Theorem 2, $TL_n(a, a-1, 1)$ is a 6-valent first-kind Frobenius circulant. By Lemma 4, and noting $\gcd(g, n) = 1$, we obtain $EJ_\alpha \cong TL_n(c, d, c+d) \cong TL_n(-cg, -dg, -(c+d)g) \cong TL_n(a, a-1, 1)$. \square

As we will see in Examples 2 and 3, a 6-valent first-kind Frobenius circulant may be isomorphic to two EJ graphs $EJ_\alpha, EJ_{\alpha'}$ with $\alpha \neq \alpha'$, and α, α' can be solutions of different equations among (6)–(8).

C. Distance distribution and examples. Given a Cayley graph Γ and integer $t \geq 0$, let $W_t(\Gamma)$ denote the number of vertices in Γ whose distance to the identity element (or any other fixed element) of the underlying group is equal to t . In §5 we will need the values of these parameters for a 6-valent first-kind Frobenius graph. Theorem 5 enables us to obtain such information by using the following known result.

Theorem 6. ([15, Theorem 27]) *Let $\alpha = c + d\rho \in \mathbb{Z}[\rho]$ be such that $\alpha \neq 0$ and $c \geq d \geq 0$. Then*

$$W_t(EJ_\alpha) = \begin{cases} 1, & t = 0 \\ 6t, & 1 \leq t < (c+d)/2 \\ 6(2c+d) - 18t, & (c+d)/2 < t < (2c+d)/3 \\ 2, & c \equiv d \pmod{3} \text{ and } t = (2c+d)/3 \\ 0, & t > (2c+d)/3. \end{cases}$$

In particular, the diameter of EJ_α is equal to $\lfloor (2c+d)/3 \rfloor$. In addition, if $c+d = 2t^$ is even, then $W_{t^*}(EJ_\alpha)$ is equal to $c^2 + cd + d^2$ minus the total number of vertices listed above.*

Theorem 6 covers all EJ graphs since any EJ graph is [15] isomorphic to some $EJ_{c+d\rho}$ with $c \geq d \geq 0$. This is because [15] $EJ_{\rho^j \alpha} \cong EJ_\alpha$ for every integer j and $EJ_{c+d\rho} \cong EJ_{d+c\rho}$. Note that we do not require $\gcd(c, d) = 1$ in Theorem 6.

We illustrate Theorems 5 and 6 by the following examples.

Example 2. Let $a \geq 3$ be an integer such that all prime factors of $n = a^2 - a + 1$ are congruent to 1 modulo 6. (Hence $a \not\equiv 2 \pmod{3}$.) Then $\Gamma = TL_n(a, a-1, 1)$ is a 6-valent first-kind Frobenius circulant. Here $k = 1$ as $n = a^2 - a + 1$, where k is as in Theorem 5. It can be verified that $(m, r, s) = (1, 0, 0)$ is a solution to each of (6)–(8). Thus, by Theorem 5, $\Gamma \cong EJ_{1-a\rho} \cong EJ_{1+(a-1)\rho} \cong EJ_{a-(a-1)\rho}$.

Regarding $\Gamma \cong EJ_{(a-1)+\rho}$ as an EJ graph allows us to compute its distance distribution. Since $a \not\equiv 2 \pmod{3}$, by Theorem 6, the diameter of Γ is $D = \lfloor (2a-1)/3 \rfloor$. Moreover, $W_t(\Gamma) = 6t$ for $1 \leq t < a/2$, and $W_t(\Gamma) = 6((2a-1) - 3t)$ for $a/2 < t \leq D$. In addition, if a is even, then $W_{a/2}(\Gamma) = n - 1 - \sum_{t \neq a/2} W_t(\Gamma) = n - 1 - \frac{3a(a-2)}{4} + (3D - \frac{3a}{2})(3D - \frac{5a}{2} + 5)$. \square

Example 3. Let $n = 12g^2 + 1$ where $g \geq 1$ is an integer. Then $v = 6g$ is a solution to $x^2 \equiv -3 \pmod{n}$ and it gives rise to the solution $a = (n + v + 1)/2 = 6g^2 + 3g + 1$ to (1)

(see Remark 1(d)). Thus $\Gamma = TL_n(6g^2 + 3g + 1, 6g^2 + 3g, 1)$ is a 6-valent first-kind Frobenius circulant.

Note that $a^2 - a + 1 = (3g^2 + 3g + 1)n$. Hence $k = 3g^2 + 3g + 1$. Since $(m, r, s) = (2g - 1, 0, g)$ is a solution to (6), by Theorem 5, $\Gamma \cong EJ_\alpha$, where $\alpha = (2g - 1) + [gn - (2g - 1)a]\rho = (2g - 1) + (2g + 1)\rho$. From (5) and the proof of Theorem 5, $\mathbb{Z}_n \rightarrow \mathbb{Z}[\rho]/(\alpha), [u] \mapsto [-u\rho]_\alpha$ defines an isomorphism from Γ to EJ_α .

It can be verified that $(m, r, s) = (2g - 1, 0, -g)$ is a solution to (7). From this we get $\Gamma \cong EJ_\beta$, where $\beta = (2g - 1) + [-gn + (2g - 1)(a - 1)]\rho = (2g - 1) - 4g\rho$, and $\mathbb{Z}_n \rightarrow \mathbb{Z}[\rho]/(\beta), [u] \mapsto [-u\rho]_\beta$ gives the required isomorphism. \square

D. EJ graphs are arc-transitive. We finish this section by proving a result which is unnoticed in the literature. Since any Cayley graph is vertex-transitive, all EJ graphs are vertex-transitive. We now prove that they are actually arc-transitive. (An arc-transitive graph without isolated vertices is vertex-transitive, but the converse is not true.) The proof is similar to that of a counterpart result [38, Lemma 7] for Gaussian graphs [26].

Theorem 7. *Let $\alpha \in \mathbb{Z}[\rho]$ be such that $N(\alpha) \geq 7$. Let*

$$H_\alpha = \{\pm[1]_\alpha, \pm[\rho]_\alpha, \pm[\rho^2]_\alpha\}.$$

Then $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ is isomorphic to a subgroup of the automorphism group of EJ_α , and EJ_α is $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ -arc-transitive.

Proof H_α is a group under the multiplication of $\mathbb{Z}[\rho]/(\alpha)$. It can be verified that

$$(x + y\rho)^{\rho^j} = (x + y\rho)\rho^j$$

defines an action of H_α on the additive group of $\mathbb{Z}[\rho]/(\alpha)$ (as a group [9]). Here and in the rest of this proof an EJ integer is interpreted as its residue class modulo α . Thus the semidirect product $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ is well-defined. Moreover, it acts on $\mathbb{Z}[\rho]/(\alpha)$ (as a set) by

$$(x + y\rho)^{(c+d\rho, \rho^j)} = ((x + c) + (y + d)\rho)\rho^j$$

for $x + y\rho \in \mathbb{Z}[\rho]/(\alpha)$ and $(c + d\rho, \rho^j) \in (\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$. It can be verified that this action is faithful, that is, the only element of $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ that fixes every $x + y\rho \in \mathbb{Z}[\rho]/(\alpha)$ is its identity element $(0, 1)$. It can also be verified that $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ preserves adjacency and non-adjacency relations of EJ_α . Hence $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(EJ_\alpha)$ of EJ_α .

Let $x_t + y_t\rho$ and $u_t + v_t\rho$ be adjacent in EJ_α , $t = 1, 2$. Then $x_t + y_t\rho = (u_t + v_t\rho) + \rho^{i_t}$ for some integer i_t . It is straightforward to verify that the element $((u_2 + v_2\rho)\rho^{i_1 - i_2} - (u_1 + v_1\rho), \rho^{i_2 - i_1})$ of $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ maps arc $(x_1 + y_1\rho, u_1 + v_1\rho)$ to arc $(x_2 + y_2\rho, u_2 + v_2\rho)$. Since this holds for any two arcs, EJ_α is $(\mathbb{Z}[\rho]/(\alpha)) \rtimes H_\alpha$ -arc-transitive. \square

4 Covers and recursive constructions

Let Γ_1 and Γ_2 be graphs. We say that Γ_1 is a *cover* of Γ_2 if there exists a surjective mapping $\phi : V(\Gamma_1) \rightarrow V(\Gamma_2)$ such that for each $u \in V(\Gamma_1)$, the restriction of ϕ to the neighbourhood $N_1(u)$ of u in Γ_1 is a bijection from $N_1(u)$ to the neighbourhood $N_2(\phi(u))$ of $\phi(u)$ in Γ_2 . If in addition $k = |\phi^{-1}(v)|$ for all $v \in V(\Gamma_2)$, then we say that Γ_1 is a *k-fold cover* of Γ_2 .

Let Γ be a graph and \mathcal{P} a partition of $V(\Gamma)$. The *quotient graph* of Γ with respect to \mathcal{P} , $\Gamma_{\mathcal{P}}$, is defined to have vertex set \mathcal{P} such that $P_1, P_2 \in \mathcal{P}$ are adjacent if and only if there exists an edge of Γ joining a vertex of P_1 to a vertex of P_2 . Let G be a group of automorphisms of Γ . If for any block $P \in \mathcal{P}$ and any $g \in G$ the image of P under g is also a block of \mathcal{P} , then \mathcal{P} is called *G-invariant*. It can be verified that, if Γ is G -arc-transitive and \mathcal{P} is G -invariant, then $\Gamma_{\mathcal{P}}$ is also G -arc-transitive.

A. Covering EJ graphs by 6-valent first-kind Frobenius circulants. We will show that any EJ graph $EJ_{c+d\rho}$ with $c, d \neq 0$ and $7 \leq N(c+d\rho) \equiv 1 \pmod{6}$ is a cover of a 6-valent first-kind Frobenius circulant. In fact, we prove the following stronger result whose proof is similar to that of [38, Lemma 8].

Theorem 8. *Let $\alpha, \beta \in \mathbb{Z}[\rho]$ be nonzero such that $N(\alpha) \geq 7$. Then $EJ_{\alpha\beta}$ is an $N(\beta)$ -fold cover of EJ_{α} and can be constructed from EJ_{α} .*

Proof Let $K = ([\alpha]_{\alpha\beta})$ be the principal ideal of $\mathbb{Z}[\rho]_{\alpha\beta}$ induced by $[\alpha]_{\alpha\beta}$. Since $\mathbb{Z}[\rho]$ is an Euclidean domain, its elements are of the form $\xi = \eta\beta + \delta$ with $\delta = 0$ or $N(\delta) < N(\beta)$. Hence $K = \{[\alpha\delta]_{\alpha\beta} : \delta \in \mathbb{Z}[\rho], \delta = 0 \text{ or } N(\delta) < N(\beta)\}$. Since $K = (\alpha)/(\alpha\beta)$, when it is viewed as a subgroup of the additive group of $\mathbb{Z}[\rho]_{\alpha\beta}$, we have $\mathbb{Z}[\rho]_{\alpha} \cong \mathbb{Z}[\rho]_{\alpha\beta}/K$ via the classical isomorphism $[\xi]_{\alpha} \mapsto K + [\xi]_{\alpha\beta}$, $[\xi]_{\alpha} \in \mathbb{Z}[\rho]_{\alpha}$. Hence $|K| = N(\alpha\beta)/N(\alpha) = N(\beta)$.

Now we construct a graph $\hat{E}J_{\alpha\beta}$ with vertex set $\mathbb{Z}[\rho]_{\alpha\beta}$ in the following way. Consider an arbitrary pair of adjacent vertices $[\xi]_{\alpha}, [\xi']_{\alpha}$ of EJ_{α} . By the definition of EJ_{α} , there exist $\eta \in \mathbb{Z}[\rho]$ and a unit ε of $\mathbb{Z}[\rho]$, both relying on ξ and ξ' , such that $\xi - \xi' = \alpha\eta + \varepsilon$. Construct $\hat{E}J_{\alpha\beta}$ in such a way that

$$\begin{aligned} \text{each } [\alpha\delta + \xi]_{\alpha\beta} \in K + [\xi]_{\alpha\beta} \text{ is adjacent to } [\alpha\delta + \xi - \varepsilon]_{\alpha\beta} = [\alpha(\delta + \eta) + \xi']_{\alpha\beta} \in K + [\xi']_{\alpha\beta} \\ \text{but not any other element in } K + [\xi']_{\alpha\beta}. \end{aligned} \quad (9)$$

This adjacency relation is defined for all pairs of adjacent vertices $[\xi]_{\alpha}, [\xi']_{\alpha}$ of EJ_{α} . Since $\xi' - \xi = -\alpha\eta - \varepsilon$, when interchanging the roles of $[\xi]_{\alpha}$ and $[\xi']_{\alpha}$ in (9), we obtain that $[\alpha\delta + \xi - \varepsilon]_{\alpha\beta} = [\alpha(\delta + \eta) + \xi']_{\alpha\beta}$ is adjacent to $[\alpha(\delta + \eta) + \xi' + \varepsilon]_{\alpha\beta} = [\alpha\delta + \xi]_{\alpha\beta}$ in $\hat{E}J_{\alpha\beta}$. Hence the adjacency relation (9) is symmetric. Moreover, it is independent of the choice of representatives of $[\xi]_{\alpha}$ and $[\alpha\delta + \xi]_{\alpha\beta}$. In fact, if $[\alpha\delta_1 + \xi_1]_{\alpha\beta} = [\alpha\delta + \xi]_{\alpha\beta}$ (which implies $[\xi_1]_{\alpha} = [\xi]_{\alpha}$), then $\xi_1 = \xi + \alpha(\sigma\beta + \delta - \delta_1)$ for some $\sigma \in \mathbb{Z}[\rho]$ and hence $\xi_1 - \xi' = \alpha(\sigma\beta + \delta - \delta_1 + \eta) + \varepsilon$. Thus, by (9), $[\alpha\delta_1 + \xi_1]_{\alpha\beta} \in K + [\xi]_{\alpha\beta}$ is adjacent to $[\alpha(\delta_1 + (\sigma\beta + \delta - \delta_1 + \eta)) + \xi']_{\alpha\beta} = [\alpha(\delta + \eta) + \xi']_{\alpha\beta} \in K + [\xi']_{\alpha\beta}$, which agrees with (9) applied to $[\alpha\delta + \xi]_{\alpha\beta}$. Therefore, $\hat{E}J_{\alpha\beta}$ is well-defined as an undirected graph. Since EJ_{α} is 6-valent, by the above construction, $\hat{E}J_{\alpha\beta}$ is 6-valent as well.

Using the notation above, by the definition of $EJ_{\alpha\beta}$, $[\alpha\delta + \xi]_{\alpha\beta}$ and $[\alpha\delta + \xi - \varepsilon]_{\alpha\beta}$ are clearly adjacent in $EJ_{\alpha\beta}$. Thus, by (9), if two vertices are adjacent in $\hat{E}J_{\alpha\beta}$, then they are adjacent in $EJ_{\alpha\beta}$. This implies that $\hat{E}J_{\alpha\beta}$ is a spanning subgraph of $EJ_{\alpha\beta}$. Since both graphs are 6-valent, it follows that they must be identical. Therefore, $EJ_{\alpha\beta}$ can be constructed from EJ_{α} as in the previous paragraph. It is obvious that the quotient graph of $EJ_{\alpha\beta}$ with respect to the partition $\mathbb{Z}[\rho]_{\alpha\beta}/K$ of $\mathbb{Z}[\rho]_{\alpha\beta}$ is isomorphic to EJ_{α} , and moreover $EJ_{\alpha\beta}$ is an $N(\beta)$ -fold cover of EJ_{α} . \square

Two elements $\alpha, \beta \in \mathbb{Z}[\rho]$ are said to be *associates* if $\alpha = \beta\rho^j$ for some integer j .

Corollary 9. *Let $\alpha = c + d\rho \in \mathbb{Z}[\rho]$ with $7 \leq N(\alpha) \equiv 1 \pmod{6}$ that is not an associate of any real integer. Denote $\ell = \gcd(c, d)$, $c' = c/\ell$, $d' = d/\ell$ and $\alpha' = c' + d'\rho$. Then EJ_{α} is an ℓ^2 -fold cover of a 6-valent first-kind Frobenius circulant that is isomorphic to $EJ_{\alpha'}$.*

Proof We have $N(\alpha) = \ell^2 N(\alpha')$ and $\ell^2 \equiv 1, 3$ or $4 \pmod{6}$ as $\ell \neq 0$. If $\ell^2 \equiv 3 \pmod{6}$, then $N(\alpha) \equiv 0$ or $3 \pmod{6}$, a contradiction. If $\ell^2 \equiv 4 \pmod{6}$, then $N(\alpha) \equiv 0, 2$ or $4 \pmod{6}$, a contradiction again. So we must have $\ell^2 \equiv 1 \pmod{6}$ and consequently $N(\alpha') \equiv 1 \pmod{6}$. We have $N(\alpha') \geq 7$, for otherwise α is an associate of the integer ℓ , a contradiction. Similarly, we have $c \neq 0$ and $d \neq 0$. Since $\gcd(c', d') = 1$, by Theorem 5 it follows that $EJ_{\alpha'}$ is isomorphic to a 6-valent Frobenius circulant. Since $\alpha = \ell\alpha'$ and $N(\ell) = \ell^2$, by Theorem 8 EJ_{α} is an ℓ^2 -fold cover of $EJ_{\alpha'}$. \square

B. Covering 6-valent first-kind Frobenius circulants. Let $n \geq 7$ be an integer and $m > 1$ a divisor of n . Let

$$K(m) = \{[km] : 0 \leq k \leq n/m - 1\}$$

be the subgroup of the additive group $(\mathbb{Z}_n, +)$ generated by $[m]$. Let

$$\mathcal{P}(m) = \mathbb{Z}_n / K(m) = \{K(m) + [j] : 0 \leq j \leq m - 1\} \cong \mathbb{Z}_m$$

be the quotient group of $(\mathbb{Z}_n, +)$ by $K(m)$. We may also view $\mathcal{P}(m)$ as a partition of \mathbb{Z}_n .

The following result states that, in some sense, any 6-valent first-kind Frobenius circulant is a cover of and can be constructed from its proper ‘quotient’ 6-valent first-kind Frobenius circulants.

Theorem 10. *Let $n \geq 7$ be an integer all of whose prime factors are congruent to 1 modulo 6. Let a be a solution to (1) and H be as in (2), so that $\Gamma = TL_n(a, a-1, 1)$ is a 6-valent first-kind Frobenius circulant of order n . Then for every proper divisor m of n , the quotient graph of Γ with respect to the partition $\mathcal{P}(m)$ is isomorphic to a 6-valent first-kind Frobenius circulant of order m , namely $\Gamma(m) = TL_m(a_m, a_m-1, 1)$, where a_m is a solution to $x^2 - x + 1 \equiv 0 \pmod{m}$. Moreover, Γ is an n/m -fold cover of $\Gamma(m)$.*

Proof Denote $H/K(m) = \{K(m) + [1], K(m) - [1], K(m) + [a], K(m) - [a], K(m) + [a-1], K(m) - [a-1]\}$. Then $-H/K(m) = H/K(m)$ and $K(m) \notin H/K(m)$ as $1, a, a-1$ are all coprime to n (Remark 1(b)). Hence $\text{Cay}(\mathcal{P}(m), H/K(m))$ is a well-defined Cayley graph. It is readily seen that $\Gamma_{\mathcal{P}(m)} \cong \text{Cay}(\mathcal{P}(m), H/K(m))$. Since $\mathcal{P}(m)$ is induced by the normal subgroup $K(m)$ of \mathbb{Z}_n , Γ must be a multcover of $\Gamma_{\mathcal{P}(m)}$; that is, for $K(m) + [j_1], K(m) + [j_2] \in \mathcal{P}(m)$ adjacent in $\Gamma_{\mathcal{P}(m)}$, every $[km + j_1] \in K(m) + [j_1]$ has the same number of neighbours in $K(m) + [j_2]$. Suppose $[j_1]$ is adjacent to distinct $[km + j_2], [k'm + j_2] \in K(m) + [j_2]$. Then $[km + j_2 - j_1], [k'm + j_2 - j_1] \in H$ and so $[(k - k')m]$ is equal to one of $\pm[1], \pm[2], \pm[a], \pm[2a], \pm[a-1], \pm[a+1], \pm[a-2], \pm[2a-1]$ and $\pm[2(a-1)]$. However, this is impossible because by Remark 1(b) all these numbers are coprime to n and hence to m . This contradiction shows that every vertex in $K(m) + [j_1]$ has exactly one neighbour in $K(m) + [j_2]$. Therefore, Γ is an n/m -fold cover of $\Gamma(m)$.

Let $a \equiv a_m \pmod{m}$, where $1 \leq a_m \leq m-1$. (Note that $a_m \neq 0$ as a and m are coprime.) Since a satisfies (1) and m divides n , a_m is a solution to $x^2 - x + 1 \equiv 0 \pmod{m}$. Let $H(m) = \langle [a]_m \rangle \leq \mathbb{Z}_m^*$, where $[x]_m$ denotes the residue class of x modulo m . By Theorem 2, $\Gamma(m) = TL_m(a_m, a_m-1, 1)$ is a 6-valent first-kind Frobenius circulant produced by $H(m)$. It is straightforward to verify that

$$\mathcal{P}(m) \rightarrow \mathbb{Z}_m, \quad K(m) + [j] \mapsto [j]_m, \quad 0 \leq j \leq m-1$$

is an isomorphism from $\Gamma_{\mathcal{P}(m)}$ to $\Gamma(m)$. \square

In Theorem 10, if the prime factorization of n is $p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$, then any $TL_n(a, a-1, 1)$ is a cover of a 6-valent first-kind Frobenius circulant of order $p_2^{e_2} \cdots p_l^{e_l}$, which in turn is a cover of a 6-valent first-kind Frobenius circulant of order $p_3^{e_3} \cdots p_l^{e_l}$, and so on. This is a property shared by the family of hypercubes.

5 Gossiping, routing and Wiener index

In this and the next sections we study gossiping (all-to-all communication), routing and broadcasting (one-to-all communication) problems for 6-valent first-kind Frobenius circulants. We will present our results in terms of such graphs, but in view of Theorem 5 the same results can also be stated in terms of EJ graphs $EJ_{c+d\rho}$ with $\gcd(c, d) = 1$ and order congruent to 1 modulo 6. At present we do not know whether the same results hold for arbitrary EJ graphs since our proofs rely on properties of Frobenius groups.

A. Routing and gossiping. A *routing* of a connected graph $\Gamma = (V, E)$ is a set of oriented paths, one for each ordered pair of vertices. The *load of an edge* with respect to a routing is the number of times it is traversed by such paths in either direction; the *load of a routing* is the maximum load on an edge; and the *edge-forward index* $\pi(\Gamma)$ is [19] the minimum load over all possible routings of Γ . The *arc-forwarding index* $\vec{\pi}(\Gamma)$ is defined [17] similarly by taking the direction into account when counting the number of times an arc is traversed. (Recall that an arc is an ordered pair of adjacent vertices.) A routing is a *shortest path routing* if all paths used are shortest paths. The *minimal edge- and arc-forwarding indices* [17], $\pi_m(\Gamma)$, $\vec{\pi}_m(\Gamma)$, are defined by restricting to shortest path routings in the definitions of π and $\vec{\pi}$, respectively. It is easy to see (e.g. [19, Theorem 3.2]) that

$$\pi_m \geq \pi \geq \frac{\sum_{(u,v) \in V \times V} d(u,v)}{|E|}, \quad \vec{\pi}_m \geq \vec{\pi} \geq \frac{\sum_{(u,v) \in V \times V} d(u,v)}{2|E|}, \quad (10)$$

where $d(u, v)$ is the distance between u and v in the graph.

An information dissemination process such that each vertex has a distinct message to be sent to all other vertices is called *gossiping* (all-to-all communication). We consider the *store-and-forward*, *all-port* and *full-duplex* model [5]: a vertex must receive a message wholly before retransmitting it to other vertices; a vertex can exchange messages (which may be different) with all of its neighbours at each time step; messages can traverse an edge in both directions simultaneously; no two messages can transmit over the same arc at the same time; and it takes one time step to transmit any message over an arc. A *gossiping scheme* is a procedure fulfilling the gossiping under these constraints, and the *minimum gossip time* [5] of a graph Γ , denoted by $t(\Gamma)$, is the minimum number of time steps required by such a scheme. Clearly, if Γ has minimum valency δ , then [5]

$$t(\Gamma) \geq \frac{|V| - 1}{\delta}. \quad (11)$$

B. Computing forwarding indices and minimum gossip time. Given a first-kind $K \rtimes H$ -Frobenius graph with diameter D , the set of vertices at distance t from the identity element of K is a union of H -orbits on K , $1 \leq t \leq D$. Denote by n_t the number of such H -orbits, and call (n_1, \dots, n_D) the *type* [13] of the graph.

In the remainder of this section, we use

$$\Gamma = TL_n(a, a-1, 1) \quad (12)$$

to denote a 6-valent first-kind Frobenius circulant, where each prime factor of $n \geq 7$ is congruent to 1 modulo 6 and a is a solution to (1). Let $D = \text{diam}(\Gamma)$ be the diameter of Γ and $\Gamma_t[0]$ the set of vertices of Γ distant t apart from $[0]$, $1 \leq t \leq D$. Then $\Gamma_t[0]$ has size $W_t(\Gamma)$. Theorems 5 and 6 together enable us to compute the type (n_1, \dots, n_D) of Γ in the following way. First, we work out $\alpha = c + d\rho$ such that $\Gamma \cong EJ_{c+d\rho}$ by using Theorem 5. Multiplying α by an appropriate ρ^j and/or interchanging c and d when necessary, we may assume $c \geq d \geq 0$. (See the paragraph right after Theorem 6.) Since each $\Gamma_t[0]$ is the union of n_t H -orbits, where $H = \langle [a] \rangle$ as in (2), we have $W_t(\Gamma) = 6n_t$ and in particular $W_t(\Gamma) \neq 2$ (hence $c \not\equiv d \pmod{3}$). Thus, by Theorem 6, $D = \lfloor (2c + d)/3 \rfloor$ and

$$n_t = \begin{cases} t, & 1 \leq t < (c + d)/2 \\ (2c + d) - 3t, & (c + d)/2 < t \leq D. \end{cases} \quad (13)$$

In addition, if $c + d = 2t^*$ is even (which can happen as seen in Example 3), then

$$6n_{t^*} = n - 1 - 6 \sum_{t \neq t^*} n_t = n - 1 - 6D(2c + d) + 9D(D + 1) + 3(c - 1)(c + d). \quad (14)$$

It is known that, for any first-kind Frobenius graph, we have $\pi = 2\vec{\pi} = 2\vec{\pi}_m = \pi_m = 2 \sum_{i=0}^D tn_i$ and these achieve the trivial lower bounds in (10) (see [13, Theorem 1.6] and [37, Theorem 6.1]). Using this and (13)–(14), we can give an explicit formula for $\pi(\Gamma) = 2\vec{\pi}(\Gamma) = 2\vec{\pi}_m(\Gamma) = \pi_m(\Gamma)$. If $c + d$ is odd, this quantity is equal to

$$D(D + 1)[(2c + d) - (2D + 1)] - \frac{1}{12}(2c - d)[(c + d)^2 - 1]; \quad (15)$$

if $c + d$ is even, it is equal to

$$\frac{1}{2}D(D + 1)[(7c + 5d) - 2(2D + 1)] + \frac{1}{12}(c + d)^2(4c + d - 6) + \frac{1}{6}[n - 3c - 5 - 6D(2c + d)]. \quad (16)$$

It is known [37, Theorem 5.1] that the minimum gossip time of any first-kind Frobenius graph achieves the trivial lower bound (11). This yields, for Γ in (12),

$$t(\Gamma) = (n - 1)/6. \quad (17)$$

In particular, for HARTS H_k (see Example 1), we get $t(H_k) = k(k - 1)/2$, which was first proved in [34, Theorem 4]. We notice that in [2, §4.3] a gossiping algorithm for H_k using $3k(k - 1)/2$ time steps was devised. We will give in Algorithm 2 an optimal gossiping algorithm for any 6-valent first-kind Frobenius circulant.

C. Geometric representation. In [37] a general method for producing optimal gossiping and routing schemes in a first-kind Frobenius graph was described. This method is abstract in nature, and it relies on knowledge of the orbits of the complement on the kernel of the underlying Frobenius group. For 6-valent first-kind Frobenius circulants, we are able to acquire such knowledge and thus give concrete optimal gossiping and routing schemes by realizing the abstract method in [37]. As we will see, the H -orbits on \mathbb{Z}_n can be visualized by using a geometric representation [35].

We label the cells of the hexagonal lattice [35] in the plane by

$$\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_6 \rightarrow \mathbb{Z}_n, (i, j, k) \mapsto [(i + ja)a^k],$$

where \mathbb{Z}_+ is the set of nonnegative integers. (See Figure 1, where, for example, $(3, 1, 0) \mapsto [(3 + 31)31^0] = [34]$ and $(2, 1, 2) \mapsto [(2 + 31)31^2] = [10]$ as $n = 49$ and $a = 31$.) The distance in Γ between $[u] \in \mathbb{Z}_n$ and $[0]$ is then given by

$$d([0], [u]) = \min \left\{ i + j : \exists (i, j, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_6, u \equiv (i + ja)a^k \pmod{n} \right\}. \quad (18)$$

Let C_ℓ be the set of hexagonal cells distant ℓ apart from a fixed $[0]$ -labelled cell in the hexagonal lattice, $\ell = 1, 2, \dots$. Then C_ℓ consists of those cells with coordinates $(i, \ell - i, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_6$, $1 \leq i \leq \ell$, $0 \leq k \leq 5$. Note that the H -orbit on \mathbb{Z}_n containing $[x] \in \mathbb{Z}_n$ is $H[x] = \{[a^i x] : i \geq 0\} = \{[a^i x] : 0 \leq i \leq 5\}$. In order to describe our optimal gossiping and routing schemes, we construct a ‘minimum distance diagram’ X by using the following algorithm.

Algorithm 1. 1. To begin with we put the six elements of $H[1]$ into X .

2. Set $\ell := 2$ and do the following:

- (a) Examine the cells $(\ell, 0, 0), (\ell - 1, 1, 0), \dots, (1, \ell - 1, 0)$ of C_ℓ one by one in this order. When examining $(i, \ell - i, 0)$, if $H[i + (\ell - i)a]$ is not contained in the current X , add all its elements to X and then move on to examine the next cell $(i - 1, \ell - i + 1, 0)$; otherwise examine the next cell straightaway.
- (b) Set $\ell := \ell + 1$ and go to Step 2(a).
- (c) Stop when all elements of $\mathbb{Z}_n \setminus \{[0]\}$ are contained in X .

In the final X each element of $\mathbb{Z}_n \setminus \{[0]\}$ appears exactly once, and X tessellates the plane [35]. See Figure 1 for $TL_{49}(31, 30, 1)$.

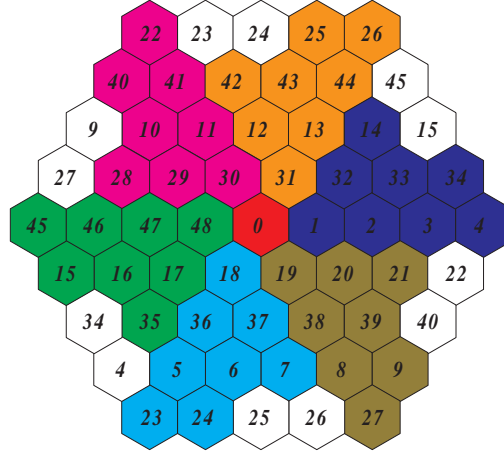


Figure 1: Hexagonal tessellation of $TL_{49}(31, 30, 1)$. The coloured area is the minimum distance diagram $X \cup \{[0]\}$, where $Y = \{[1], [2], [3], [4], [32], [33], [34], [14]\}$ is the part of X in the first sector. The other five sectors $Y[31], Y[30], -Y, -Y[31], -Y[30]$ of X are obtained by rotating Y about the origin by $60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$ respectively. This graph has diameter 4 and type $(i_0, i_1, i_2) = (4, 3, 1)$.

Slightly abusing terminology, we may take X as the set of cells $(i, \ell - i, k)$ such that $[(i + (\ell - i)a)a^k] \in X$. The shape of X is determined by the values of the parameter i_j defined as follows. Let

$$r = \max \{i \geq 1 : (i, 0, 0) \text{ is contained in } X\}.$$

Then $d([0], [i]) = i$, $0 \leq i \leq r$, for otherwise $r = d([0], [r]) \leq d([0], [i]) + d([i], [r]) = d([0], [i]) + d([0], [r - i]) \leq (i - 1) + (r - i)$. On the other hand, for any $i \geq r + 1$, $d([0], [i]) = d([0], [ia]) \leq i - 1$

by the definition of r . Thus, if $j \geq r+1$ and $i \geq 1$, then $(i, j, 0)$ is not contained in X for otherwise $i+j = d([0], [i+ja]) \leq d([0], [ja]) + d([ja], [i+ja]) = d([0], [ja]) + d([0], [i]) \leq (j-1) + i$. Define

$$i_j = \max \{i \geq 0 : (i, j, 0) \text{ is contained in } X\}, \quad 0 \leq j \leq r.$$

Then $i_0 = r$ and i_j is well-defined as $d([0], [ja]) = d([0], [j]) = j$ and so $(0, j, 0)$ belongs to X . The values of i_j can be obtained by running Algorithm 1.

Denote by Y the subset of X in the first sector of the hexagonal lattice.

Lemma 11. *With the notation above, the following hold:*

- (a) $Y = \{[i+ja] : 1 \leq i \leq i_j, 0 \leq j \leq r\}$, $X = \cup_{k=0}^5 Y[a^k] = \{[(i+ja)a^k] : 1 \leq i \leq i_j, 0 \leq j \leq r, 0 \leq k \leq 5\}$, and every element of $\mathbb{Z}_n \setminus \{[0]\}$ appears in X exactly once;
- (b) if $[i+ja] \in Y$, then $d([0], [(i+ja)a^k]) = i+j$, $0 \leq k \leq 5$;
- (c) $\sum_{j=0}^r i_j = (n-1)/6 \geq r = i_0 \geq i_1 \geq \dots \geq i_r \geq 0$;
- (d) $D = \max\{i_j + j : 0 \leq j \leq r\}$;
- (e) $n_t = |\Gamma_t[0] \cap Y| = |\{[i+ja] \in Y : i+j = t\}| = |\{j : 0 \leq j \leq r, i_j + j \geq t\}|$, $1 \leq t \leq D$.

Proof (a) It suffices to prove that, if $i_j \geq 1$ for some $0 \leq j \leq r$, then $d([0], [i+ja]) = i+j$ for every $1 \leq i \leq i_j$. Suppose otherwise. Then $i_j + j = d([0], [i_j+ja]) \leq d([0], [i+ja]) + d([i+ja], [i_j+ja]) = d([0], [i+ja]) + d([0], [i_j-i]) \leq (i+j-1) + (i_j-i) = i_j + j - 1$, a contradiction.

(c) Suppose $i_{j-1} < i_j$ for some j . Then $(i_{j-1} + 1) + (j-1) \geq d([0], [(i_{j-1} + 1) + (j-1)a]) \geq d([0], [i_j+ja]) - d([(i_{j-1} + 1) + (j-1)a], [i_j+ja]) = d([0], [i_j+ja]) - d([0], [(i_j - i_{j-1} - 1) + a]) \geq (i_j + j) - (i_j - i_{j-1}) = (i_{j-1} + 1) + (j-1)$. Thus $d([0], [(i_{j-1} + 1) + (j-1)a]) = (i_{j-1} + 1) + (j-1)$, which contradicts the definition of i_{j-1} . So we have $i_{j-1} \geq i_j$ for $1 \leq j \leq r$. The truth of $\sum_{j=0}^r i_j = (n-1)/6$ follows from (a) and the symmetry of X .

The truth of (b), (d) and (e) follows from (a) and the definition of X . \square

Part (a) of Lemma 11 implies that X is partitioned into six sectors, namely $Y, Y[a], Y[a^2] = Y[a-1], Y[a^3] = -Y, Y[a^4] = -Y[a], Y[a^5] = -Y[a-1]$, which are permuted cyclically by H .

D. Optimal routing and gossiping schemes. Guided by the general approach in [37], we now construct a spanning tree T_0 of Γ rooted at $[0]$ and use it to give optimal gossiping and routing in Γ . Let $A_{1,1} = \{([0], [v]) : [v] \in H\}$ and add these six arcs to T_0 . Inductively, for $0 \leq t \leq D-1$ and each $[v_l] \in \Gamma_{t+1}[0] \cap Y$ ($1 \leq l \leq n_{t+1}$), choose a neighbour $[u_l]$ of $[v_l]$ in $\Gamma_t[0] \cap Y$ and add arcs $A_{t+1,l} = \{([u_l]a^k, [v_l]a^k) : 0 \leq k \leq 5\}$ to T_0 . (It is allowed to have $u_l = u_{l'}$ for $l \neq l'$.) Thus the branches of T_0 in $Y[a^k]$ are obtained by rotating the branch of T_0 in Y by $(60k)^\circ$ and the set of arcs of T_0 from $T_0(t)$ to $T_0(t+1)$ is $\cup_{1 \leq l \leq n_{t+1}} A_{t+1,l}$, where $T_0(t)$ is the set of vertices distant t apart from $[0]$ in T_0 . Since H is semiregular on $\mathbb{Z}_n \setminus \{[0]\}$, one can show that each $A_{t+1,l}$ is a matching of six arcs (see [37]). Note that $T_0(t) = \Gamma_t[0]$, $0 \leq t \leq D$, and T_0 is a *shortest path spanning tree* of Γ with root $[0]$, that is, the unique path in T_0 between $[0]$ and any vertex is a shortest path in Γ .

For $[u] \in \mathbb{Z}_n$, define T_u to be the graph with vertex set \mathbb{Z}_n and arcs $([x+u], [y+u])$ with $([x], [y])$ running over all arcs of T_0 . Since \mathbb{Z}_n acts on itself (by addition) as a group of automorphisms of Γ , T_u is a shortest path spanning tree of Γ with root $[u]$. Denote by P_{uv} the unique path in T_u from $[u]$ to $[v]$. Define

$$\mathcal{P} = \{P_{uv} : [u], [v] \in \mathbb{Z}_n, [u] \neq [v]\}. \quad (19)$$

Algorithm 2. Let M_u denote the message originating at $[u] \in \mathbb{Z}_n$.

PHASE 1: Initially, M_u is transmitted from $[u]$ to $T_0(1) + [u]$ along the six arcs of $A_{1,1} + [u]$, and this is carried out for all $[u] \in \mathbb{Z}_n$ simultaneously.

PHASE $t + 1$: Do the following for $t = 1, 2, \dots, D - 1$ successively: for $l = 1, 2, \dots, n_{t+1}$, in the l th step of the $(t + 1)$ th phase, for all $[u] \in \mathbb{Z}_n$ transmit M_u from $T_0(t) + [u]$ to $T_0(t + 1) + [u]$ along the six arcs of $A_{t+1,l} + [u]$ at the same time step.

A routing \mathcal{P} of Γ is called G -arc-transitive [25] for some $G \leq \text{Aut}(\Gamma)$ if every element of G maps paths of \mathcal{P} to paths of \mathcal{P} and moreover G is transitive on the set of arcs of Γ . A routing under which all edges (arcs, respectively) have the same load is called *edge-uniform* (*arc-uniform*, respectively).

The following is a consequence of Theorem 2, Lemma 11, [13, Theorem 1.6] and [37, Theorems 5.1 and 6.1].

Corollary 12. Let $\Gamma = TL_n(a, a - 1, 1)$ be a 6-valent first-kind Frobenius circulant, where each prime factor of $n \geq 7$ is congruent to 1 modulo 6 and a is a solution to (1). Then $\pi(\Gamma) = 2\vec{\pi}(\Gamma) = 2\vec{\pi}_m(\Gamma) = \pi_m(\Gamma)$ and it is given by (15) or (16) (depending on whether the corresponding $c + d$ is odd or even), and $t(\Gamma)$ is given by (17). Moreover, \mathcal{P} given in (19) is a shortest path routing of Γ which is $\mathbb{Z}_n \rtimes H$ -arc transitive (where H is as given in (2)), edge- and arc-uniform, and optimal for π , $\vec{\pi}$, $\vec{\pi}_m$ and π_m simultaneously.

Furthermore, Algorithm 2 gives an optimal gossiping scheme for Γ such that: (a) the message originating from any vertex is transmitted along shortest paths to other vertices; (b) for each vertex $[w]$ of Γ , at any time precisely six arcs are used to transmit the message originating from $[w]$, and at any time ≥ 2 these six arcs form a matching of Γ ; (c) at any time each arc of Γ is used exactly once for message transmission.

We remark that the spanning tree T_0 constructed above is not unique, and different choices of T_0 produce different optimal gossiping and routing schemes.

As mentioned earlier, Γ achieves the trivial lower bounds in (10). Using this and Lemma 11, we obtain a second formula for $\pi(\Gamma)$:

$$\pi(\Gamma) = \sum_{j=0}^r i_j(i_j + 2j + 1). \quad (20)$$

Example 4. It can be verified that, for $\Gamma = TL_{n_k}(3k + 2, 3k + 1, 1)$ ($k \geq 2$) in Example 1, we have $i_j = k - j$ ($0 \leq j \leq k - 1$) and $i_k = 0$. From this and (20) we recover the result $\pi(\Gamma) = k(k + 1)(2k + 1)/2$ obtained in [34, Theorem 5]. In [34], the authors also gave optimal routing and gossiping schemes for this particular graph. Corollary 12 generalizes these to all 6-valent first-kind Frobenius circulants.

The graph Γ in Example 3 satisfies $i_j = 2g - j$ ($0 \leq j \leq g - 1$), $i_j = 2g - j - 1$ ($g \leq j \leq 2g - 1$) and $i_{2g} = 0$. From this we obtain $\pi(\Gamma) = 2g(8g^2 + 1)/3$ by (20) and $t(\Gamma) = 2g^2$ by (17). \square

E. Wiener index. The *Wiener index* of a graph is the sum of the distances between all unordered pairs of vertices. With motivation from chemistry, this index has attracted considerable interest in chemical graph theory over sixty years (see [10] for a survey on the topic for hexagonal systems). As a by-product of the discussion above, we obtain the following result.

Corollary 13. The Wiener index of any 6-valent first-kind Frobenius circulant $TL_n(a, a - 1, 1)$ is equal to $3n/2$ times the expression in (15) or (16), depending on whether the corresponding $c + d$ is odd or even, or equivalently $3n/2$ times the right-hand side of (20).

6 Broadcasting

A process of disseminating a message from a *source vertex* x to all other vertices in a network Γ is called *broadcasting* [20] (one-to-all communication) if in each time step any vertex who has received the message already can retransmit it to at most one of its neighbours. Let $b(\Gamma, x)$ be the minimum t such that all vertices receive the message after t steps. The *broadcasting time* [20] of Γ , denoted by $b(\Gamma)$, is the maximum among $b(\Gamma, x)$ for x running over all vertices of Γ .

Since the diameter is a trivial lower bound on the broadcasting time, any graph whose broadcasting time is close to its diameter may be thought as efficient in terms of broadcasting. The following result shows that all 6-valent first-kind Frobenius circulants are such graphs.

Theorem 14. *Let $\Gamma = TL_n(a, a-1, 1)$ be a 6-valent first-kind Frobenius circulant, where each prime factor of $n \geq 7$ is congruent to 1 modulo 6 and a is a solution to (1). Let D be the diameter of Γ . Then*

$$b(\Gamma) = D + 2 \text{ or } D + 3 \quad (21)$$

and both $D + 2$ and $D + 3$ are attainable. In particular, if $n = 12g^2 + 1 \geq 49$ and $a = 6g^2 + 3g + 1$ as in Example 3, then

$$b(\Gamma) = D + 3 = 2g + 3. \quad (22)$$

Proof We use the notation and results in the previous section. Since Γ is vertex-transitive, it suffices to prove $D + 2 \leq b(\Gamma, [0]) \leq D + 3$.

We prove the lower bound first. Suppose to the contrary that $b(\Gamma, [0]) \leq D + 1$. Then there exists a broadcasting scheme for Γ using $D + 1$ time steps. Let M denote the message at $[0]$ to be broadcasted to other vertices. At time 1 the message is sent from $[0]$ to exactly one of the six vertices of H . At time 2 the message can be sent to at most two vertices of H . So at least three vertices of H receive M at time 3 or later. Hence there exists k such that both $[a^k]$ and $[a^{k+1}]$ receive M at time 3 or later. Since X is symmetric, there exists a vertex $[ua^k] \in Y[a^k]$ whose distance to $[0]$ in Γ is equal to D . Since any shortest path from $[0]$ to $[ua^k]$ has to use $[a^k]$ or $[a^{k+1}]$, and since these two vertices receive M at time 3 or later, $[ua^k]$ receives M at time $D + 2$ or later, contradicting our assumption. Therefore, $b(\Gamma, [0]) \geq D + 2$.

We prove the upper bound by giving a broadcasting scheme explicitly. Before doing so let us explain our notation first. A broadcasting scheme with source vertex $[0]$ can be defined by specifying a pair $L(x) = (t_x, y_x)$ for each $x \neq [0]$, which means that x receives the message at time t_x from a neighbour y_x of x . We require $t_y < t_x$ for $y = y_x$ and $(t_x, y_x) \neq (t_z, y_z)$ if $x \neq z$.

Using the notation above, we define

$$L([1]) = (1, [0]), L([a]) = (2, [1]), L([a^3]) = (2, [0])$$

$$L([a^2]) = (3, [a]), L([a^4]) = (3, [a^3]), L([a^5]) = (3, [0]).$$

For each $k = 0, 1, \dots, 5$, define

$$L([ia^k]) = (i + 2, [(i - 1)a^k]), \text{ for } 2 \leq i \leq r$$

$$L([(i + ja)a^k]) = (i + j + 3, [(i + (j - 1)a)a^k]), \text{ for } 1 \leq i \leq r - 1, j \geq 1, [i + ja] \in Y$$

$$L([(r + ja)a^k]) = (r + j + 2, [(r + (j - 1)a)a^k]), \text{ for } j \geq 1, [r + ja] \in Y.$$

It is straightforward (but laborious) to verify that this L defines a broadcasting scheme for Γ . Since the maximum value of $i + j$ such that $[i + ja] \in Y$ is equal to D , this broadcasting can be completed in at most $D + 3$ time steps. Therefore, $b(\Gamma, [0]) \leq D + 3$. Moreover, if there is

only one vertex $[u]$ in Y such that $d([0], [u]) = D$ and further $[u]$ is of the form $[r + ja]$ for some $j \geq 0$, then L requires only $D + 2$ time steps and hence $b(\Gamma) = D + 2$. This occurs when, for example, $\Gamma = TL_{43}(7, 6, 1) \cong EJ_{7-6\rho}$. Thus the lower bound in (21) is attainable.

We now prove (22) for Γ with $n = 12g^2 + 1 \geq 49$ and $a = 6g^2 + 3g + 1$ (see Example 3). As mentioned in Example 4, this special graph Γ satisfies $i_j = 2g - j$ ($0 \leq j \leq g - 1$), $i_j = 2g - j - 1$ ($g \leq j \leq 2g - 1$) and $i_{2g} = 0$. Hence its diameter $D = 2g$. It suffices to prove $b(\Gamma, [0]) \geq 2g + 3$.

Suppose to the contrary that there exists a broadcasting scheme for Γ using $2g + 2$ time steps. By Lemma 11(a), we have $d([0], [(2g - j) + ja]a^k) = 2g$ for $0 \leq j \leq g - 1$ and $0 \leq k \leq 5$. In particular, $d([0], [2ga^k]) = 2g$ and $P_k : [0], [a^k], [2a^k], \dots, [2ga^k]$ is the unique shortest path from $[0]$ to $[2ga^k]$. Among the six vertices of H , exactly one receives M at time 1. If at most one vertex of H receives M at time 2, then at least one vertex in H , say, $[a^k]$, receives M at time 4. Since P_k has length $2g$, this implies that $[2ga^k]$ receives M at time $2g + 3$ or later, which contradicts our assumption. Thus exactly two vertices of H receive M at time 2, and the remaining three vertices receive M at time 3 or later. However, if a vertex in H , say, $[1]$, receives M at time 4 or later, then since P_0 is the unique path from $[0]$ to $[2g]$, $[2g]$ receives M at time $2g + 3$ or later, which is a contradiction. Therefore the times that the vertices of H (in cyclic order) receive M must be $(1, 2, 2, 3, 3, 3)$, $(1, 2, 3, 2, 3, 3)$, $(1, 2, 3, 3, 2, 3)$ or $(1, 2, 3, 3, 3, 2)$. In each case there are two consecutive vertices of H (with respect to the cyclic order) which receive M at time 3. Without loss of generality we may assume that $[1]$ and $[a]$ receive M at time 3. Since the broadcasting finishes in $2g + 2$ time steps, and since P_0 is the unique shortest path from $[0]$ to $[2g]$ and its length is $2g$, each vertex $[i]$ on P_0 has to receive M from $[i - 1]$ at time $i + 2$, $2 \leq i \leq 2g$. Similarly, each vertex $[ia]$ on P_1 receives M from $[(i - 1)a]$ at time $i + 2$, $2 \leq i \leq 2g$. Since $d([0], [(2g - 1) + a]) = d$, $[(2g - 1) + a]$ has to receive M via a shortest path Q from $[0]$ to $[2g - 1]$. Note that Q uses either $[1]$ or $[a]$ as its second vertex. In the former case, Q is of the form $[0], [1], \dots, [s], [s + a], [s + a + 1], \dots, [(2g - 1) + a]$ for some $1 \leq s \leq 2g - 1$. However, since $[s]$ sends M to $[s + 1]$ at time $s + 3$, it cannot send M to $[s + a]$ at time $s + 3$. Thus $[s + a]$ receives M from $[s]$ at time $s + 4$ or later. Consequently, $[s + a + 1]$ receives M from $[s + a]$ at time $s + 5$ or later, and so on. Finally, $[(2g - 1) + a]$ receives M from $[(2g - 2) + a]$ at time $2g + 3$ or later, contradicting our assumption. In the case when Q uses $[a]$, it is the path $[0], [a], [a + 1], \dots, [a + (2g - 1)]$. Since $[a]$ sends M to $[2a]$ at time 4, it cannot send M to $[a + 1]$ at time 4. Hence $[a + 1]$ receives M from $[a]$ at time 5 or later, and so on, and $[a + (2g - 1)]$ receives M at time $2g + 3$ or later, which is again a contradiction. Therefore, $b(\Gamma[0]) \geq 2g + 3$ and (22) holds for the graph in Example 3. \square

We notice that for HARTS H_k (see Example 1) an optimal broadcasting algorithm was given in [6, Algorithm A2], which uses $k + 2$ steps if $k \geq 3$ and $k + 1 = 3$ steps if $k = 2$. Since H_k is a 6-valent first-kind Frobenius circulant of diameter $k - 1$ (Example 1), this result is now a special case of Theorem 14.

7 Concluding remarks

In this paper we classified all 6-valent first-kind Frobenius circulants and studied gossiping, routing and broadcasting in them. Such graphs are efficient for routing and gossiping in the sense that they have the smallest possible forwarding indices and gossip time under the store-and-forward, all-port and full-duplex model. We gave optimal gossiping and routing schemes for them by utilizing the method in [37]. We proved that the broadcasting time of any 6-valent first-kind Frobenius circulant is equal to its diameter plus 2 or 3, with both values achievable. This

indicates that such graphs are also efficient for broadcasting. A by-product of our discussion on routing is a formula for the Wiener index of any 6-valent first-kind Frobenius circulant.

We proved that 6-valent first-kind Frobenius circulants are precisely EJ graphs $EJ_{c+d\rho}$ with $\gcd(c, d) = 1$ and order congruent to 1 modulo 6. We further proved that any EJ graph with order congruent to 1 modulo 6 and the corresponding EJ integer not an associate of a real integer, is a topological cover of a 6-valent first-kind Frobenius circulant. This is a consequence of a stronger result which asserts that, roughly, larger EJ graphs can be constructed from smaller ones as topological covers. A similar result was proved for the family of 6-valent first-kind Frobenius circulants. We also proved that all EJ graphs are arc-transitive. An interesting question arisen from our study is whether non-Frobenius EJ graphs are as efficient as 6-valent first-kind Frobenius circulants in terms of routing, gossiping and broadcasting, and whether they also have the smallest possible forwarding indices and gossip time.

In Example 1 we saw that, for any integer $k \geq 2$, $TL_{n_k}(3k+2, 3k+1, 1)$ is a 6-valent first-kind Frobenius circulant with $n_k = 3k^2 + 3k + 1$ vertices. This graph has the maximum possible order [35] among all 6-valent geometric circulants of diameter k , and when $k = 2$ is the HARTS network physically tested at the University of Michigan as a distributed real-time computing system. Note that n_k can be a composite number (e.g. $n_5 = 91 = 7 \cdot 13$), and in this case by (e) of Theorem 2, besides $TL_{n_k}(3k+2, 3k+1, 1)$ there is at least one more 6-valent first-kind Frobenius circulant of order n_k . It would be interesting to explicitly construct all of them.

Finally, various combinatorial properties of 6-valent first-kind Frobenius circulants, and EJ graphs in general, deserve further investigation. The results in [28] imply that the chromatic number of any 6-valent first-kind Frobenius circulant Γ of order $n \equiv 1 \pmod{6}$ is equal to 4 if $n \neq 7, 13, 19$, and in these exceptional cases Γ has chromatic number 7, 5, 5, respectively. Thus, if $n > 19$, then the independence number of Γ is at least $\lceil n/4 \rceil$. At present we do not know whether this bound is sharp in general.

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